2 Existence of infinitesimal isometries on Riemannian manifolds of dimension 2

Let (M, g) be a smooth Riemannian manifold of dimension n. A smooth vector field ξ on M is an infinitesimal isometry (or a Killing field) if and only if ξ satisfies

$$L_\xi g = 0, \tag{2.1}$$

where L is the Lie derivative. In terms of local coordinates $x = (x^1, \cdots, x^n)$ (2.1) becomes

$$\xi^\lambda_i g_{\lambda j} + \xi^\lambda_j g_{\lambda i} - \xi^\lambda g_{ij,\lambda} = 0, \quad i, j = 1, \cdots, n, \tag{2.2}$$

where $g_{ij} = g(\partial_i, \partial_j)$ and $\xi = \xi^\lambda \frac{\partial}{\partial x^\lambda}$ (summation convention for $\lambda = 1, \cdots, n$). Since (2.2) is symmetric in (i, j) the number of equations in (2.2) is $\frac{n(n+1)}{2}$ whereas the number of unknowns is n so that (2.2) is overdetermined if $n \geq 2$. In this section we shall present a coordinate-free computation of prolongation of (2.1) with $n = 2$ to a complete system of order 2 and discuss the existence of solutions. Let $\{e_1, e_2\}$ be an orthonormal frame over a 2-dimensional Riemannian manifold M and let ω^1, ω^2 be the dual coframe. Then

$$g = \omega^1 \circ \omega^1 + \omega^2 \circ \omega^2,$$

where $\phi \circ \eta := \frac{1}{2}(\phi \otimes \eta + \eta \otimes \phi)$ is the symmetric product of 1-forms. Recall also that there exist a uniquely determined 1-form ω^1_2 (Levi-Civita connection) and a function K (Gaussian curvature) satisfying

$$\left.\begin{array}{rl} d\omega^1 = & -\omega^1_2 \wedge \omega^2 \\ d\omega^2 = & \omega^1_2 \wedge \omega^1 \end{array}\right\} \tag{2.3}$$

and

$$d\omega^1_2 = K\omega^1 \wedge \omega^2. \tag{2.4}$$

Furthermore, Lie derivatives of ω^i, $i = 1, 2$ with respect to a vector field $\xi = \xi^1 e_1 + \xi^2 e_2$ are

$$\begin{array}{rcl} L_\xi \omega^1 & = & d(\xi \lrcorner \omega^1) + \xi \lrcorner d\omega^1 \\ & = & d\xi^1 - \omega^1_2(\xi)\omega^2 + \xi^2 \omega^1_2 \ \text{ by (2.3)} \end{array} \tag{2.5}$$

and similarly

$$\begin{array}{rcl} L_\xi \omega^2 & = & d(\xi \lrcorner \omega^2) + \xi \lrcorner d\omega^2 \\ & = & d\xi^2 + \omega^1_2(\xi)\omega^1 - \xi^1 \omega^1_2. \end{array} \tag{2.6}$$

By (2.5) and (2.6), we have

$$\begin{aligned}\frac{1}{2}L_\xi g &= (L_\xi\omega^1)\circ\omega^1 + (L_\xi\omega^2)\circ\omega^2\\ &= (d\xi^1 + \xi^2\omega_2^1)\circ\omega^1 + (d\xi^2 - \xi^1\omega_2^1)\circ\omega^2.\end{aligned}$$

By setting

$$\left\{\begin{aligned} d\xi^1 &= -\xi^2\omega_2^1 + \xi_1^1\omega^1 + \xi_2^1\omega^2,\\ d\xi^2 &= \xi^1\omega_2^1 + \xi_1^2\omega^1 + \xi_2^2\omega^2\end{aligned}\right. \tag{2.7}$$

and substituting in the above we have

$$\frac{1}{2}L_\xi g = \xi_1^1\omega^1\circ\omega^1 + (\xi_2^1 + \xi_1^2)\omega^1\circ\omega^2 + \xi_2^2\omega^2\otimes\omega^2.$$

By (2.1), ξ is an infinitesimal isometry if and only if

$$\xi_1^1 = \xi_2^2 = 0,\ \ \xi_2^1 + \xi_1^2 = 0. \tag{2.8}$$

Substituting (2.8) in (2.7) we see that a vector field $\xi = \xi^1 e_1 + \xi^2 e_2$ is an infinitesimal isometry if and only if

$$\left\{\begin{aligned} d\xi^1 &= -\xi^2\omega_2^1 + \xi_2^1\omega^2,\\ d\xi^2 &= \xi^1\omega_2^1 + \xi_1^2\omega^1,\end{aligned}\right. \tag{2.9}$$

which is a coordinate-free version of (2.2) with $n = 2$ expressed as an exterior differential system. Prolongation of (2.9) to a complete system is differentiating (2.9) and expressing $(d\xi^1, d\xi^2, d\xi_2^1)$ in terms of (ξ^1, ξ^2, ξ_2^1) :
We apply d to (2.9) and substitute (2.9), (2.3) and (2.4) for $d\xi^i$, $d\omega^i$ and $d\omega_2^1$, respectively, to obtain

$$\begin{aligned}(d\xi_2^1 - K\xi^2\omega^1)\wedge\omega^2 = 0,\\ (d\xi_2^1 + K\xi^1\omega^2)\wedge\omega^1 = 0.\end{aligned}$$

Hence we have

$$d\xi_2^1 = K(\xi^2\omega^1 - \xi^1\omega^2). \tag{2.10}$$

The system (2.9) and (2.10) is a prolongation of (2.1) to a complete system. Now consider the Euclidean space $\mathbb{R}^3$ of variables (ξ^1, ξ^2, ξ_2^1). Then the submanifold of the first jet space of ξ defined by (2.8) may be identified with

$\mathcal{S} := M \times \mathbb{R}^3$.
On $M \times \mathbb{R}^3$ consider the Pfaffian system $\theta = (\theta^1, \theta^2, \theta^3)$ given by

$$\begin{array}{rcl} \theta^1 & = & d\xi^1 + \xi^2\omega_2^1 - \xi_2^1\omega^2, \\ \theta^2 & = & d\xi^2 - \xi^1\omega_2^1 + \xi_2^1\omega^1, \\ \theta^3 & = & d\xi_2^1 - K\xi^2\omega^1 + K\xi^1\omega^2. \end{array} \tag{2.11}$$

We check the Frobenius integrability conditions for (2.11): By (2.3) and (2.4) we have

$$d\theta^1, d\theta^2 \equiv 0 \qquad \mod \theta$$

and

$$d\theta^3 \equiv (K_1\xi^1 + K_2\xi^2)\omega^1 \wedge \omega^2 \qquad \mod \theta$$

where $K_i = dK(e_i),\ \ i = 1, 2$ so that $dK = K_1\omega^1 + K_2\omega^2$.
Thus (2.11) is integrable if and only if $T := K_1\xi^1 + K_2\xi^2$ is identically zero on $M \times \mathbb{R}^3$, which is equivalent to $K_1 = K_2 = 0$ i.e. K is constant. In this case, there exist 3 parameter family of solutions by the Frobenius theorem. Otherwise, assuming $dT \neq 0$ on $T = 0$, we consider a submanifold $\mathcal{S}'$ of dimension 4 defined by $T = 0$.
Differentiating $dK = K_1\omega^1 + K_2\omega^2$, we see by (2.3) that

$$\begin{array}{rcl} 0 & = & d^2K \\ & = & (dK_1 + K_2\omega_2^1)\omega^1 + (dK_2 - K_1\omega_2^1)\omega^2. \end{array} \tag{2.12}$$

Thus we put

$$dK_1 = -K_2\omega_2^1 + K_{11}\omega^1 + K_{12}\omega^2, \tag{2.13}$$
$$dK_2 = K_1\omega_2^1 + K_{21}\omega^1 + K_{22}\omega^2. \tag{2.14}$$

By substituting (2.13), (2.14) in (2.12) we have $K_{12} = K_{21}$.

On $\mathcal{S}'$, we have by (2.11), (2.13) and (2.14)

$$\begin{array}{rcl} dT & = & \xi^1 dK_1 + K_1 d\xi^1 + \xi^2 dK_2 + K_2 d\xi^2 \\ & \equiv & (K_{11}\xi^1 + K_{12}\xi^2 - K_2\xi_2^1)\omega^1 + (K_{12}\xi^1 + K_{22}\xi^2 + K_1\xi_2^1)\omega^2 \qquad \mod \theta. \end{array}$$

We set

$$\left\{ \begin{array}{rcl} T_1 & = & K_{11}\xi^1 + K_{12}\xi^2 - K_2\xi_2^1, \\ T_2 & = & K_{12}\xi^1 + K_{22}\xi^2 + K_1\xi_2^1. \end{array} \right. \tag{2.15}$$

If $T_1, T_2 \equiv 0$ on $\mathcal{S}'$, $i^*\theta^1, i^*\theta^3, i^*\theta^3$ have rank 2 by Theorem 1.1. Then $\mathcal{S}'$ is foliated by two dimensional integral manifolds and therefore there are 2 parameter family of solutions. But this implies that $K_1 = K_2 = 0$ which is impossible.

Let $A = \begin{pmatrix} K_1 & K_2 & 0 \\ K_{11} & K_{12} & -K_2 \\ K_{12} & K_{22} & K_1 \end{pmatrix}$.

If $\det A = 0$, A has rank 2 and $\mathcal{S}'' = \{T = T_1 = T_2 = 0\}$ is a 3-dimensional submanifold of $\mathcal{S}$. If we have $dT_1, dT_2 \equiv 0 \mod \theta^1, \theta^2, \theta^3$ on $\mathcal{S}''$, Theorem 1.1 and the Frobenius theorem imply that $\mathcal{S}''$ is foliated by two dimensional integral manifolds and therefore there exists 1 parameter family of solutions. To calculate dT_1, dT_2 we differentiate (2.13). Then we have

$$\begin{aligned} 0 &= d^2K_1 \\ &= (dK_{11} + 2K_{12}\omega_2^1 + K_2K\omega^2)\omega^1 + (dK_{12} + K_{22}\omega_2^1 - K_{11}\omega_2^1)\omega^2. \end{aligned} \tag{2.16}$$

Thus we put

$$dK_{11} = -2K_{12}\omega_2^1 + K_{111}\omega^1 + K_{112}\omega^2, \tag{2.17}$$
$$dK_{12} = (K_{11} - K_{22})\omega_2^1 + K_{121}\omega^1 + K_{122}\omega^2. \tag{2.18}$$

By substituting (2.17), (2.18) in (2.16) we have $K_{112} = K_{121} - K_2K$.

Differentiating (2.14), we have

$$\begin{aligned} 0 &= d^2K_2 \\ &= (dK_{12} + K_{22}\omega_2^1 - K_{11}\omega_2^1)\omega^1 + (dK_{22} - 2K_{12}\omega_2^1 + K_1K\omega^1)\omega^2. \end{aligned} \tag{2.19}$$

By substituting (2.17), (2.18) in (2.19) we have

$$(dK_{22} - 2K_{12}\omega_2^1 + K_1K\omega^1 - K_{122}\omega^1)\omega^2 = 0.$$

Thus we put

$$dK_{22} = 2K_{12}\omega_2^1 + (K_{122} - K_1K)\omega^1 + K_{222}\omega^2. \tag{2.20}$$

On $\mathcal{S}''$, we have by (2.11), (2.17), (2.18) and (2.20)

$$\begin{aligned} dT_1 \quad &\equiv \quad (K_{111}\xi^1 + (K_{121} - K_2K)\xi^2 - 2K_{12}\xi^1_2)\omega^1 \\ &\quad +(K_{121}\xi^1 + K_{122}\xi^2 + (K_{11} - K_{22})\xi^1_2)\omega^2 \qquad \mod \ \theta \end{aligned}$$

and

$$\begin{aligned} dT_2 \quad &\equiv \quad (K_{121}\xi^1 + K_{122}\xi^2 + (K_{11} - K_{22})\xi^1_2)\omega^1 \\ &\quad +((K_{122} - K_1K)\xi^1 + K_{222}\xi^2 + 2K_{12}\xi^1_2)\omega^2 \qquad \mod \ \theta. \end{aligned}$$

We summarize the discussions of this section in the following

Theorem 2.1 *Let M be a Riemannian manifold of dimension 2.*

Let $\mathbf{K} = \begin{pmatrix} K_1 & K_2 & 0 \\ K_{11} & K_{12} & -K_2 \\ K_{12} & K_{22} & K_1 \\ K_{111} & K_{121} - K_2K & -2K_{12} \\ K_{121} & K_{122} & K_{11} - K_{22} \\ K_{122} - K_1K & K_{222} & 2K_{12} \end{pmatrix}.$

(i) If $\mathbf{K}$ has rank 0, there exist 3 parameter family of infinitesimal isometries,

(ii) If $\mathbf{K}$ has rank 2 and $(K_1, K_2) \neq 0$, there exist 1 parameter family of infinitesimal isometries,

(iii) If $\mathbf{K}$ has rank 3, there exists only trivial infinitesimal isometry.

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